

On 2- and 3-periodic Lyness difference equations

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We describe the sequences $\{x_n\}_n$ given by the non-autonomous second-order Lyness difference equations $x_{n+2} = (a_n + x_{n+1})/x_n$, where $\{a_n\}_n$ is either a 2-periodic or a 3-periodic sequence of positive values and the initial conditions x_1, x_2 are also positive. We also show an interesting phenomenon of the discrete dynamical systems associated with some of these difference equations: the existence of one oscillation of their associated rotation number functions. This behaviour does not appear for the autonomous Lyness difference equations.

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1. Introduction and main result

This paper fully describes the sequences given by the non-autonomous second-order Lyness difference equations

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \quad (1)$$

where $\{a_n\}_n$ is a k -periodic sequence taking positive values, $k = 2, 3$ and the initial conditions x_1, x_2 are as well positive. This question is proposed in ([5], Section 5). Recall that non-autonomous recurrences appear, for instance, as population models with a variable structure affected by some seasonality [11,12], where k is the number of seasons. Some dynamical issues of similar type of equations have been studied in several recent papers [1,9,10,14–16,18].

Recall that when $k = 1$, that is $a_n = a > 0$, for all $n \in \mathbb{N}$, then (1) is the famous Lyness recurrence which is well understood, see for instance [2,19]. The cases $k = 2, 3$ have been already studied and some partial results are established. For both cases, it is known that the solutions are persistent near a given k -periodic solution, which is stable. This is proved by using some known invariants, see [15,17,18]. Recall that in our context, it is said that a solution $\{x_n\}_n$ is persistent if there exist two real positive constants c and C , which depend on the initial conditions, such that for all $n \geq 1$, $0 < c < x_n < C < \infty$.

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We prove:

THEOREM 1. *Let $\{x_n\}_n$ be any sequence defined by (1) and $k \in \{2, 3\}$. Then, it is persistent. Furthermore, either,*

- (a) *The sequence $\{x_n\}_n$ is periodic, with period a multiple of k , or*
- (b) *the sequence $\{x_n\}_n$ densely fills one or two (resp. one, two or three) disjoint intervals of \mathbb{R}^+ when $\{a_n\}_n$ is 2-periodic (resp. 3-periodic). Moreover, it is possible by algebraic tools to distinguish which is the situation.*

Our approach to describe the sequences $\{x_n\}_n$ is based on the study of the natural dynamical system associated with (1) and on the results of [7]. The main tool that allows to distinguish the number of intervals for the adherence of the sequences $\{x_n\}_n$ is the computation of several resultants, see Section 4.

It is worth to comment that Theorem 1 is an extension of what happens in the classical case $k = 1$. There, the same result holds but in statement (b) only one interval appears. Our second main result will prove that there are other more significant differences between the case $k = 1$ and the cases $k = 2, 3$. These differences are related to the lack of monotonicity of certain rotation number functions associated with the dynamical systems given by the Lyness recurrences, see Theorem 3. The behaviours of these rotation number functions are important for the understanding of the recurrences because they give the possible periods for them, see [2,4,19].

On the other hand in [10,15], it is proved that, at least for some values of $\{a_n\}_n$, the behaviour of $\{x_n\}_n$ for the case $k = 5$ is totally different. In particular, unbounded positive solutions appear. In the forthcoming paper [8], we explore in more detail the differences between the cases $k = 1, 2, 3$ and $k \geq 4$.

This paper is organized as follows: Section 2 presents the difference equations that we are studying as discrete dynamical systems and we state our main results of them, see Theorems 2 and 3. Section 3 is devoted to the Proof of Theorem 2. Using it in Section 4, we prove Theorem 1 and we give some examples of how to apply it to determine the number of closed intervals of the adherence of $\{x_n\}_n$. In Section 5, we demonstrate Theorem 3 and we also present some examples where we study in more detail the rotation number function of the dynamical systems associated with (1).

2. Main results from the dynamical systems point of view

In this section, we reduce the study of the sequence $\{x_n\}_n$ to the study of some discrete dynamical systems and we state our main results of them.

First, we introduce some notations. When $k = 2$, set

$$a_n = \begin{cases} a & \text{for } n = 2\ell + 1, \\ b & \text{for } n = 2\ell, \end{cases} \quad (2)$$

and when $k = 3$, set

$$a_n = \begin{cases} a & \text{for } n = 3\ell + 1, \\ b & \text{for } n = 3\ell + 2, \\ c & \text{for } n = 3\ell, \end{cases} \quad (3)$$

where $\ell \in \mathbb{N}$ and $a > 0, b > 0$ and $c > 0$.

We also consider the maps $F_\alpha(x, y)$, with $\alpha \in \{a, b, c\}$, as

$$F_\alpha(x, y) = \left(y, \frac{\alpha + y}{x} \right),$$

defined on the open invariant set $Q^+ := \{(x, y) : x > 0, y > 0\} \subset \mathbb{R}^2$.

Consider for instance $k = 2$. The sequence given by (1),

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, \dots, \quad (4)$$

can be seen as

$$(x_1, x_2) \xrightarrow{F_a} (x_2, x_3) \xrightarrow{F_b} (x_3, x_4) \xrightarrow{F_a} (x_4, x_5) \xrightarrow{F_b} (x_5, x_6) \xrightarrow{F_a} \dots$$

Hence, the behaviour of (4) can be obtained from the study of the dynamical system defined in Q^+ by the map:

$$F_{b,a}(x, y) := F_b \circ F_a(x, y) = \left(\frac{a + y}{x}, \frac{a + bx + y}{xy} \right).$$

Similarly, for $k = 3$, we can consider the map:

$$F_{c,b,a}(x, y) := F_c \circ F_b \circ F_a(x, y) = \left(\frac{a + bx + y}{xy}, \frac{a + bx + y + cxy}{y(a + y)} \right).$$

It can be proved that both maps have an only fixed point in Q^+ , which depends on a, b (and c), which for short we denote by \mathbf{p} .

It is easy to interpret the invariants for (1) and $k = 2, 3$, given in [16,18], in terms of first integrals of the above maps, see also Lemma 6. We have that

$$V_{b,a}(x, y) := \frac{ax^2y + bxy^2 + bx^2 + ay^2 + (b^2 + a)x + (b + a^2)y + ab}{xy},$$

is a first integral for $F_{b,a}$ and

$$V_{c,b,a}(x, y) := \frac{cx^2y + axy^2 + bx^2 + by^2 + (a + bc)x + (c + ab)y + ac}{xy},$$

is a first integral for $F_{c,b,a}$. Notice that

$$abV_{b,a}(x, y) + a^2b^2 + a^3 + b^3 = \frac{(bx + a)(ay + b)(ax + by + ab)}{xy},$$

and

$$abcV_{c,b,a}(x, y) + a^2b^2 + a^2c^2 + b^2c^2 = \frac{(bcx + aby + ac)(acxy + abx + bcy + abc)}{xy},$$

are as well first integrals of $F_{b,a}$ and $F_{c,b,a}$, respectively. They remain the well-known expression

$$V_a(x, y) = \frac{(x + 1)(y + 1)(x + y + a)}{xy},$$

of the first integral of Lyness' map F_a .

The topology of the level sets of $V_{b,a}$ and $V_{c,b,a}$ in Q^+ as well as the dynamics of the maps restricted to them is described by the following result, which will be proved in Section 3.

THEOREM 2.

- (i) *Each level set of $V_{b,a}$ (resp. $V_{c,b,a}$) in $Q^+ \setminus \{\mathbf{p}\}$ is diffeomorphic to a circle surrounding \mathbf{p} , which is the unique fixed point of $F_{b,a}$ (resp. $F_{c,b,a}$).*
- (ii) *The action of $F_{b,a}$ (resp. $F_{c,b,a}$) on each level set of $V_{b,a}$ (resp. $V_{c,b,a}$) contained in $Q^+ \setminus \{\mathbf{p}\}$ is conjugated to a rotation of the circle.*

Once a result like the above one is established, the study of the possible periods of the sequences $\{x_n\}_n$ given by (1) is quite standard. It suffices first to get the rotation interval, which is the open interval formed by all the rotation numbers given by the above theorem, varying the level sets of the first integrals. Afterwards, it suffices to find which are the denominators of all the irreducible rational numbers that belong to the corresponding interval, see [4,6,19].

The study of the rotation number of these kinds of rational maps is not an easy task, see again [2,4,6,19]. In particular, in [4], it was proved that the rotation number function parameterized by the energy levels of Lyness map $F_{a,a} \neq 1$ is always monotonous, solving a conjecture of Zeeman given in [19], see also [13]. As far as we know, in this paper, we give the first simple example for which this rotation number function is neither constant nor monotonous. We prove:

THEOREM 3. *There are positive values of a and b , such that the rotation number function $\rho_{b,a}(h)$ of $F_{b,a}$ associated with the closed ovals of $\{V_{b,a} = h\} \subset Q^+$ has a local maximum.*

Hence, apart from the known behaviours for the autonomous Lyness maps, that is global periodicity or monotonicity of the rotation number function, which trivially holds for $F_{b,a}$, taking for instance $a = b = 1$ or $a = b \neq 1$, respectively, there appear more complicated behaviours for the rotation number function.

Our proof of this result relies on the study of lower and upper bounds for the rotation number of $F_{b,a}$ on a given oval of a level set of $V_{b,a}$ given for some $(a, b) \in Q^+$ and $\{V_{b,a}(x, y) = V_{b,a}(x_0, y_0)\}$, for $(x_0, y_0) \in Q^+$. This can be done because the map on this oval is conjugated to a rotation and it is possible to use an algebraic manipulator to follow and to order a finite number iterates on it, which are also given by points with rational coordinates. So, only exact arithmetic is used. A similar study could be done for $F_{c,b,a}$.

3. Proof of Theorem 2

Proof of (i) of Theorem 2. Case $k = 2$. The orbits of $F_{b,a}$ lie on the level sets of $V_{b,a}$. First, we prove that if $h > 0$ the sets $\{(x, y) \in Q^+ : V_{b,a}(x, y) \leq h\}$ are compact. From the inequality

$$V_{b,a}(x, y) = ax + by + b\frac{x}{y} + a\frac{y}{x} + \frac{b^2 + a}{y} + \frac{b + a^2}{x} + \frac{ab}{xy} \leq h,$$

we get that

$$\frac{b + a^2}{h} \leq x \leq \frac{h}{a} \quad \text{and} \quad \frac{a + b^2}{h} \leq y \leq \frac{h}{b}.$$

Notice that this result in particular already implies the persistence of the sequences given by (1).

Next, let us show that in Q^+ , the set of fixed points of $F_{b,a}$ and the set of singular points of $V_{b,a}$ coincide and they only contain a point \mathbf{p} . Indeed, the fixed points of $F_{b,a}$ are given by $x^2 = a + y$ and $y^2 = b + x$. Since $f(x) := a + \sqrt{b + x} - x^2$ is concave, $f(0) > 0$, $f'(0) > 0$ and $\lim_{x \rightarrow +\infty} f(x) = -\infty$, there is a unique positive zero of f , and therefore there is a unique fixed point \mathbf{p} in Q^+ .

Now, notice that

$$\frac{\partial V_{b,a}(x,y)}{\partial x} = \frac{ay + b}{x^2 y} (x^2 - y - a) \quad \text{and} \quad \frac{\partial V_{b,a}(x,y)}{\partial y} = \frac{bx + a}{xy^2} (y^2 - x - b).$$

So in Q^+ the only fixed point of $F_{b,a}$ coincides with the only singular point of $V_{b,a}$, which necessarily is the point where it attains its absolute minimum.

The above facts prove that each level set of $V_{b,a}$ in Q^+ is, for $h > h_c := V_{b,a}(\mathbf{p})$, diffeomorphic to a circle: it is compact, a differentiable closed curve without singular points and the boundary of the connected open set $\{V_{b,a}(x,y) < h\}$ from statement (5) of ([3], Prop. 2.1).

Case $k = 3$. By using the same kind of arguments as in the previous case, the result follows. We will only give the details of the proof that the fixed point of $F_{c,b,a}$ in Q^+ is unique and coincides with the unique singular point of $V_{c,b,a}$ in Q^+ .

Clearly, a fixed point in Q^+ satisfies

$$y = g(x) := \frac{bx + a}{x^2 - 1} \quad \text{and} \quad y(y + a) = x(x + c). \quad (5)$$

So the fixed point in Q^+ is given by the unique intersection point in $\{x > 1\}$ of the graphs of the decreasing function $y = g(x)$ and the increasing one $y = (-a + \sqrt{a^2 + 4x(x + c)})/2$.

The point (x, y) is a singular point of $V_{c,b,a}$ if and only if

$$x^2 = \frac{(by + c)(y + a)}{cy + b} \quad \text{and} \quad y^2 = \frac{(bx + a)(x + c)}{ax + b}. \quad (6)$$

Consider $\Delta := x(x + c) - y(y + a)$. By taking $x + c$ from the right-hand side equation (6) and $y + a$ from the left one, we obtain that

$$\Delta = \frac{y(ax + b)(by + c) - x(bx + a)(cy + b)}{(bx + a)(by + c)} xy.$$

Some computations, by using once more equation (6), show that

$$\begin{aligned} & y(ax + b)(by + c) - x(bx + a)(cy + b) \\ &= b(axy^2 + by^2 + cy - cx^2y - bx^2 - ax) = b(y^2(ax + b) + cy - x^2(cy + b) - ax) \\ &= b[(bx + a)(x + c) + cy - (by + c)(y + a) - ax] = b^2[x(x + c) - y(y + a)]. \end{aligned}$$

Therefore,

$$\Delta = \frac{b^2xy}{(bx+a)(by+c)}\Delta.$$

If $\Delta \neq 0$, we obtain that $aby + cbx + ac = 0$, which is impossible. So $\Delta = 0$, which means that any singular point verifies the second equation of the fixed point.

Combining the right-hand side equations of (5) and (6), we get

$$ay = (x+c)\left(x - \frac{bx+a}{ax+b}\right) \quad \text{or equivalently} \quad \frac{y}{x+c} = \frac{x^2-1}{ax+b}.$$

By using again the right-hand side equation of (6), we get that $y = (bx+a)/(x^2-1)$ as we wanted to prove. \square

3.1 Proof of (ii) of Theorem 2

In [7], a result is proved that characterizes the dynamics of integrable diffeomorphisms having a *Lie symmetry*, that is a vector field \mathbf{X} such that $\mathbf{X}(F(p)) = (DF(p))\mathbf{X}(p)$. Next theorem states it, particularized to the case we are interested.

THEOREM 4 (CIMA ET AL. [7]). *Let $\mathcal{U} \subset \mathbb{R}^2$ be an open set and let $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ be a diffeomorphism such that*

- (a) *It has a smooth regular first integral $V : \mathcal{U} \rightarrow \mathbb{R}$, having its level sets $\Gamma_h := \{z = (x, y) \in \mathcal{U} : V(z) = h\}$ as simple closed curves.*
- (b) *There exists a smooth function $\mu : \mathcal{U} \rightarrow \mathbb{R}^+$ such that for any $z \in \mathcal{U}$,*

$$\mu(\Phi(z)) = \det(D\Phi(z))\mu(z).$$

Then, the map Φ restricted to each Γ_h is conjugated to a rotation with rotation number $\tau(h)/T(h)$, where $T(h)$ is the period of Γ_h as a periodic orbit of the planar differential equation

$$\dot{z} = \mu(z) \left(-\frac{\partial V(z)}{\partial y}, \frac{\partial V(z)}{\partial x} \right),$$

and $\tau(h)$ is the time needed by the flow of this equation for going from any $w \in \Gamma_h$ to $\Phi(w) \in \Gamma_h$.

Next lemma is one of the key points for finding a Lie symmetry for families of periodic maps, like the 2- and 3-periodic Lyness maps.

LEMMA 5. *Let $\{G_a\}_{a \in A}$ be a family of diffeomorphisms of $\mathcal{U} \subset \mathbb{R}^2$. Suppose that there exists a smooth map $\mu : \mathcal{U} \rightarrow \mathbb{R}$ such that for any $a \in A$ and any $z \in \mathcal{U}$, the equation $\mu(G_a(z)) = \det(DG_a(z))\mu(z)$ is satisfied. Then, for every choice $a_1, \dots, a_k \in A$, we have*

$$\mu(G_{[k]}(z)) = \det(DG_{[k]}(z))\mu(z),$$

where $G_{[k]} = G_{a_k} \circ \dots \circ G_{a_2} \circ G_{a_1}$.

Proof. It is only necessary to prove the result for $k = 2$, because the general case follows easily by induction. Consider $a_1, a_2 \in A$, then

$$\begin{aligned}\mu(G_{a_2, a_1}(z)) &= \mu(G_{a_2} \circ G_{a_1}(z)) = \det(DG_{a_2}(G_{a_1}(z)))\mu(G_{a_1}(z)) \\ &= \det(DG_{a_2}(G_{a_1}(z)))\det(DG_{a_1}(z))\mu(z) = \det(D(G_{a_2} \circ G_{a_1}(z)))\mu(z) \\ &= \det(DG_{a_2, a_1}(z))\mu(z),\end{aligned}$$

and the lemma follows. \square

Proof of (ii) of Theorem 2. From part (i) of the theorem, we know that the level sets of $V_{b,a}$ and $V_{c,b,a}$ in $Q^+ \setminus \{\mathbf{p}\}$ are diffeomorphic to circles. Moreover, these functions are first integrals of $F_{b,a}$ and $F_{c,b,a}$, respectively. Notice also that for any a , the Lyness map $F_a(x, y) = (y, (a + y/x))$ satisfies

$$\mu(F_a(x, y)) = \det(DF_a(x, y))\mu(x, y),$$

with $\mu(x, y) = xy$. Hence, by Lemma 5,

$$\mu(F_{b,a}(x, y)) = \det(DF_{b,a}(x, y))\mu(x, y) \quad \text{and} \quad \mu(F_{c,b,a}(x, y)) = \det(DF_{c,b,a}(x, y))\mu(x, y).$$

Thus, from Theorem 4, the result follows. \square

It is worth to comment that once part (i) of the theorem is proved, it is also possible to prove that the dynamics of $F_{b,a}$ (resp. $F_{c,b,a}$) restricted to the level sets of $V_{b,a}$ (resp. $V_{c,b,a}$) is conjugated to a rotation by using that they are given by cubic curves and that the map is birational, see [17]. We prefer our approach because it provides a dynamical interpretation of the rotation number together with its analytic characterization.

4. Proof of Theorem 1

In order to prove Theorem 1, we need a preliminary result. Consider the maps $F_{b,a}$ and $F_{a,b}$, jointly with their corresponding first integrals $V_{b,a}$ and $V_{a,b}$. In a similar way, consider $F_{c,b,a}$, $F_{a,c,b}$ and $F_{b,a,c}$ with $V_{c,b,a}$, $V_{a,c,b}$ and $V_{b,a,c}$. Some simple computations prove the following elementary but useful lemma. Notice that it can be interpreted as the relation between the first integrals and the non-autonomous invariants.

LEMMA 6. *With the above notations:*

- (i) $V_{b,a}(x, y) = V_{a,b}(F_a(x, y))$.
- (ii) $V_{c,b,a}(x, y) = V_{a,c,b}(F_a(x, y)) = V_{b,a,c}(F_b(F_a(x, y)))$.

Proof of Theorem 1. We split the proof in two steps. For $k = 2, 3$ we first prove that there are only two types of behaviours for $\{x_n\}_n$, either this set of points is formed by kp points for some positive integer p , or it has infinitely many points whose adherence is given by at most k intervals. Secondly, in the latter case, we provide an algebraic way for studying the actual number of intervals.

First step: We start with the case $k = 2$. With the notation introduced in (2), it holds that

$$F_{b,a}(x_{2n-1}, x_{2n}) = (x_{2n+1}, x_{2n+2}), \quad F_{a,b}(x_{2n}, x_{2n+1}) = (x_{2n+2}, x_{2n+3}),$$

where $(x_1, x_2) \in Q^+$ and $n \geq 1$. So the odd terms of the sequence $\{x_n\}_n$ are contained in the projection on the x -axis of the oval of $\{V_{b,a}(x, y) = V_{b,a}(x_1, x_2) = h\}$ and the even terms in the corresponding projection of $\{V_{a,b}(x, y) = V_{a,b}(F_a(x_1, x_2)) = V_{b,a}(x_1, x_2) = h\}$, where notice that we have used Lemma 6.

Recall that the ovals of $V_{b,a}$ are invariant by $F_{b,a}$ and the ovals of $V_{a,b}$ are invariant by $F_{a,b}$. Notice also that the trivial equality $F_a \circ F_{b,a} = F_{a,b} \circ F_a$ implies that the action of $F_{b,a}$ on $\{V_{b,a}(x, y) = h\}$ is conjugated to the action of $F_{a,b}$ on $\{V_{a,b}(x, y) = h\}$ via F_a .

From Theorem 2, we know that $F_{b,a}$ on the corresponding oval is conjugated to a rotation of the circle. Hence, if the corresponding rotation number is rational, then the orbit starting at (x_1, x_2) is periodic, of period say q , then the sequence $\{x_n\}_n$ is $2q$ -periodic. On the other hand, if the rotation number is irrational, then the orbit of (x_1, x_2) generated by $F_{b,a}$ fulfils densely the oval of $\{V_{b,a}(x, y) = h\}$ in Q^+ and hence the subsequence of odd terms also fulfils densely the projection of $\{V_{b,a}(x, y) = h\}$ in the x -axis. Clearly, the sequence of even terms does the same with the projection of the oval of $\{V_{a,b}(x, y) = h\}$.

Similarly, when $k = 3$, the equalities

$$\begin{aligned} F_{c,b,a}(x_{3n-2}, x_{3n-1}) &= (x_{3n+1}, x_{3n+2}), \\ F_{a,c,b}(x_{3n-1}, x_{3n}) &= (x_{3n+2}, x_{3n+3}), \\ F_{b,a,c}(x_{3n}, x_{3n+1}) &= (x_{3n+3}, x_{3n+4}), \end{aligned}$$

where $n \geq 1$, allow to conclude that each term x_m , of the sequence $\{x_n\}_n$ where we use the notation (3), is contained in one of the projections on the x -axis of the ovals $\{V_{c,b,a}(x, y) = V_{c,b,a}(x_1, x_2) = h\}$ and $\{V_{a,c,b}(x, y) = h\}$ and $\{V_{b,a,c}(x, y) = h\}$, according to the remainder of m after dividing it by 3. The rest of the proof in this case follows as in the case $k = 2$. So the first step is done.

Second step: from the above results, it is clear that the problem of knowing the number of connected components of the adherence of $\{x_n\}_n$ is equivalent to the control of the projections of several invariant ovals on the x -axis. The strategy for $k = 3$, and analogously for $k = 2$, is the following. Consider the ovals contained in the level sets given by $\{V_{c,b,a}(x, y) = h\}$, $\{V_{a,c,b}(x, y) = h\}$, and $\{V_{b,a,c}(x, y) = h\}$ and denoted by $I = I(a, b, c, h)$, $J = J(a, b, c, h)$ and $K = K(a, b, c, h)$ and the corresponding closed intervals of $(0, \infty)$ given by their projections on the x -axis.

We want to detect the values of h for which two of the intervals, among I , J and K , have exactly one common point. First, we seek their boundaries. Since the level sets are given by cubic curves, which are quadratic with respect to the y -variable, these points will correspond to the values of x for which the discriminant of the quadratic equation with respect to y is zero. So we compute

$$\begin{aligned} R_1(x, h, a, b, c) &:= \text{dis}(xyV_{c,b,a}(x, y) - hxy, y) = 0, \\ R_2(x, h, a, b, c) &:= \text{dis}(xyV_{a,c,b}(x, y) - hxy, y) = 0, \\ R_3(x, h, a, b, c) &:= \text{dis}(xyV_{b,a,c}(x, y) - hxy, y) = 0. \end{aligned}$$

Now we have to search for relations among a, b, c and h for which two of these three functions have some common solution, x . These relations can be obtained by computing some suitable resultants.

Taking the resultants of R_1 and R_2 , R_2 and R_3 , and R_1 and R_3 with respect to x , we obtain three polynomial equations $R_4(h, a, b, c) = 0$, $R_5(h, a, b, c) = 0$ and $R_6(h, a, b, c) = 0$. In short, once a, b and c are fixed, we have obtained three polynomials in h such that a subset

of their zeroes give the bifurcation values which separate the number of intervals of the adherence of $\{x_n\}_n$. See the results of Proposition 7 and Example 8 for concrete applications of the method.

Before ending the proof, we want to comment that for most values of $a-c$, varying h there appear the three possibilities, namely 1, 2 or 3 different intervals. The last case appears for values of h near $h_c := V_{c,b,a}(\mathbf{p})$, because the first coordinates of the three points \mathbf{p} , $F_a(\mathbf{p})$ and $F_b(F_a(\mathbf{p}))$ almost never coincide. The other situations can be obtained by increasing h . \square

PROPOSITION 7. *Consider the recurrence (1) with $k = 2$ and $\{a_n\}_n$ as in (2) taking the values $a = 3$ and $b = 1/2$. Define $h_c = (12z^3 - 33z + 7)/(2(z^2 - 3)) \simeq 17.0394$, where $z \simeq 2.1513$ is the biggest positive real root of $2z^4 - 12z^2 - 2z + 17$ and $h^* \simeq 17.1198$ is the smallest positive root of*

$$p_4(h) := 112900h^4 - 2548088h^3 - 48390204h^2 + 564028596h + 7613699255.$$

Then,

- (i) *The initial condition $(x_1, x_2) = (z, z^2 - 3)$ gives a 2-periodic recurrence $\{x_n\}_n$. Moreover, $V_{1/2,3}(z, 3 - z^2) = h_c$.*
- (ii) *Let (x_1, x_2) be any positive initial condition, different from $(z, z^2 - 3)$ and set $h = V_{1/2,3}(x_1, x_2)$. Let $\rho(h)$ denote the rotation number of $F_{1/2,3}$ restricted to the oval of $\{V_{1/2,3}(x, y) = h\}$. Then,*
 - (I) *If $\rho(h) = p/q \in \mathbb{Q}$, with $\gcd(p, q) = 1$, then the sequence $\{x_n\}_n$ is $2q$ -periodic.*
 - (II) *If $\rho(h) \notin \mathbb{Q}$ and $h \in (h_c, h^*)$, then the adherence of the sequence $\{x_n\}_n$ is formed by two disjoint closed intervals.*
 - (III) *If $\rho(h) \notin \mathbb{Q}$ and $h \in [h^*, \infty)$, then the adherence of the sequence $\{x_n\}_n$ is one closed interval.*

We want to remark that, from a computational point of view, the case (I) is almost never detected. Indeed, taking a and b rational numbers and starting with rational initial conditions, by almost using Mazur's theorem, it can be seen that the rotation number will almost never be rational, see the proof of ([2], Prop. 1). Therefore, in numeric simulations, only situations (II) and (III) appear, and the value $h = h^*$ gives the boundary between them. In general, for $k = 2$, the value h^* is always the root of a polynomial of degree 4, which is constructed from the values of a and b .

Proof of Proposition 7. Clearly $(z, 3 - z^2)$ is the fixed point of $F_{b,a}$ in Q^+ . Some computations give the compact expression of $h_c := V_{a,b}(z, z^2 - 3)$. To obtain the values h^* , we proceed as in the Proof of Theorem 1. In general,

$$\begin{aligned} R_1(x, h, a, b) &:= \text{dis}(xyV_{b,a}(x, y) - hxy, y) \\ &= (ax^2 - hx + a^2 + b)^2 - 4(bx + a)(bx^2 + b^2x + ax + ab), \\ R_2(x, h, a, b) &:= \text{dis}(xyV_{a,b}(x, y) - hxy, y) \\ &= (bx^2 - hx + a + b^2)^2 - 4(ax + b)(ax^2 + a^2x + bx + ab). \end{aligned}$$

Then we have to compute the resultant of the above polynomials with respect to x . It always decomposes as the product of two quartic polynomials in h . Its expression is very

large, so we only give it when $a = 3$ and $b = 1/2$. It is written as

$$\frac{625}{65536}(4h^4 - 1176h^3 + 308h^2 + 287380h + 1816975)p_4(h).$$

It has four real roots, two for each polynomial. Some further work proves that the one that interests us is the smallest one of p_4 . \square

We also give an example when $k = 3$ but skipping all the details.

Example 8. Consider the recurrence (1) with $k = 3$ and $\{a_n\}_n$ as in (3) taking the values $a = 1/2, b = 2$ and $c = 3$. Then, for any positive initial conditions x_1 and x_2 , $V_{c,b,a}(x_1, x_2) = h \geq V_{c,b,a}(\mathbf{p}) = h_c \approx 15.9283$. Moreover, if the rotation number of $F_{c,b,a}$ associated with the oval $\{V_{c,b,a}(x, y) = h\}$ is irrational, then the adherence of $\{x_n\}_n$ is given by

- Three intervals when $h \in (h_c, h^*)$, where $h^* \approx 15.9614$;
- Two intervals when $h \in [h^*, h^{**})$, where $h^{**} \approx 16.0015$;
- One interval when $h \in [h^{**}, \infty)$.

The values h^* and h^{**} are roots of two polynomials of degree 8 with integer coefficients that can be explicitly given.

Finally, for $k = 2$, we give an easy sufficient condition to ensure that the projections of the level sets of $\{V_{b,a}(x, y) = h\}$ and $\{V_{a,b}(x, y) = h\}$ on the x -axis are not disjoint.

Remark 9. Consider the recurrence (1) with $k = 2$ and $\{a_n\}_n$ as in (2). Let $\mathbf{p}_{b,a}$ and $\mathbf{p}_{a,b}$ be the fixed points of $F_{b,a}$ and $F_{a,b}$ in Q^+ , respectively. Then, if either $V_{b,a}(\mathbf{p}_{a,b}) < V_{b,a}(x_1, x_2)$ or $V_{a,b}(\mathbf{p}_{b,a}) < V_{a,b}(x_2, x_3)$, the sequence $\{x_n\}$ is periodic or densely fills exactly one interval of \mathbb{R}^+ .

5. Some properties of the rotation number function

From Theorem 2, it is natural to introduce the *rotation number function* for $F_{b,a}$:

$$\rho_{b,a} : [h_c, \infty) \rightarrow (0, 1),$$

where $h_c := V_{b,a}(\mathbf{p})$, as the map that associates with each invariant oval $\{V_{b,a}(x, y) = h\}$, the rotation number $\rho_{b,a}(h)$ of the function $F_{b,a}$ is restricted to it. The following properties hold:

- (i) The function $\rho_{b,a}(h)$ is analytic for $a > 0, b > 0, h > h_c$ and it is continuous at $h = h_c$. This can be proved from the tools introduced in ([6], Section 4).
- (ii) The value $\rho_{b,a}(h_c)$ is given by the argument over 2π of the eigenvalues (which have modulus 1 due to the integrability of $F_{b,a}$) of the differential of $F_{b,a}$ at \mathbf{p} .
- (iii) $\rho_{b,a}(h) = \rho_{a,b}(h)$ and
- (iv) $\rho_{a,a}(h) = 2\rho_a(h) \bmod 1$, where ρ_a is the rotation number³ function associated with the classical Lyness map. Then, from the results of [4], we know that $\rho_{1,1}(h) \equiv 3/5$, that for $a \neq 1$, positive, $\rho_{a,a}(h)$ is monotonous and $\lim_{h \rightarrow \infty} \rho_{a,a}(h) = 3/5$.

Note that item (iii) follows because $F_{a,b}$ is conjugated with $F_{b,a}$ via $\psi = F_a$ which is a diffeomorphism of Q^+ , because $\psi^{-1}F_{a,b}\psi = F_a^{-1}F_aF_bF_a = F_bF_a = F_{b,a}$. Since ψ preserves the orientation, the rotation number functions of $F_{a,b}$ and $F_{b,a}$ restricted to the corresponding ovals must coincide.

Similar results to the ones given above hold for $F_{c,b,a}$ and its corresponding rotation number function. Some obvious differences are

$$\begin{aligned}\rho_{c,b,a}(h) &= \rho_{b,a,c}(h) = \rho_{a,c,b}(h), & \rho_{a,a,a}(h) &= 3\rho_a(h) \pmod{1}, \\ \rho_{1,1,1}(h) &= 2/5, & \lim_{h \rightarrow \infty} \rho_{a,a,a}(h) &= 2/5.\end{aligned}$$

We are convinced that when $a > 0$ and $b > 0$,

$$\lim_{h \rightarrow \infty} \rho_{b,a}(h) = 3/5 \quad \text{and} \quad \lim_{h \rightarrow \infty} \rho_{c,b,a}(h) = 2/5,$$

but we have not been able to prove these equalities. If they were true, by combining them with the values of the rotation number function at $h = h_c$, this would give a very useful information to decide if, apart from the trivial cases $a = b = 1$ ($c = 1$), there are other cases for which the rotation number function is constant. Notice that in these situations the maps $F_{b,a}$ or $F_{c,b,a}$ would be globally periodic in Q^+ . This information, together with the values at h_c , also would be useful to know the regions where the corresponding functions could be increasing or decreasing. Finally, notice that this rotation number at infinity is not continuous when we approach to $a = 0$ or $b = 0$, where the recurrence and the first integral are also well defined on Q^+ . For instance, $\rho_{0,0}(h) \equiv 2/3$ and the numerical experiments of the next subsection seem to indicate that for $a > 0$ or $b > 0$,

$$\lim_{h \rightarrow \infty} \rho_{0,a}(h) = \lim_{h \rightarrow \infty} \rho_{b,0}(h) = 5/8.$$

Before proving Theorem 3, we introduce with an example the algorithm that we will use along this section to compute lower and upper bounds for the rotation number. We have implemented it in an algebraic manipulator. Notice also that when we apply it taking rational values of a and b and rational initial conditions, it can be used as a method to achieve proofs, see next example or the Proof of Theorem 3.

Fix $a = 3$, $b = 2$ and $(x_0, y_0) = (1, 1)$. Then, $h = V_{2,3}(1, 1) = 34$. Compute, for instance, the 27 points of the orbit starting at $(1, 1)$,

$$(x_1, y_1) = (4, 6), \quad (x_2, y_2) = \left(\frac{9}{4}, \frac{17}{24}\right), \quad (x_3, y_3) = \left(\frac{89}{54}, \frac{788}{153}\right), \dots$$

and consider them as points on the oval $\{V_{2,3}(x, y) = 34\}$ (see Figure 1).

We already know that the restriction of $F_{2,3}$ to the given oval is conjugated to a rotation, with rotation number $\rho := \rho_{2,3}(34)$ that we want to estimate. This can be done by counting the number of turns that give the points (x_j, y_j) , after fixing some orientation in the closed curve. We orient the curve with the counter-clockwise sense. So, for instance, we know that the second point (x_2, y_2) has given more than one turn and less than two, giving that $1 < 2\rho < 2$, and hence that $\rho \in (1/2, 1)$. Doing the same reasoning with all

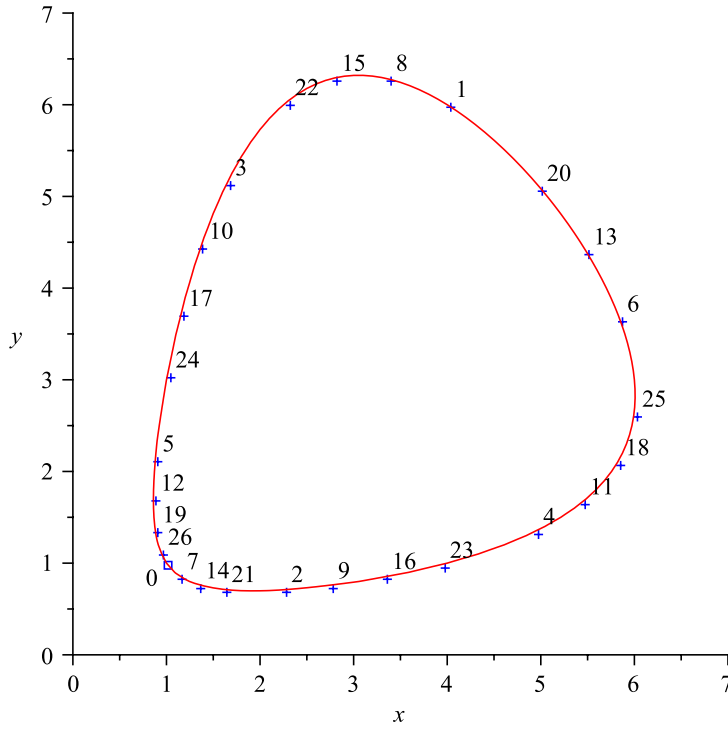


Figure 1. Oval of $\{V_{2,3}(x,y) = 34\}$ with 27 iterates of $F_{2,3}$. The label 0 indicates the initial condition $(1, 1)$, and the label $k, k = 1, \dots, 26$, corresponds to the k -th point of the orbit.

the points computed, we obtain

$$\begin{aligned} 4 < 7\rho < 5 &\rightarrow \rho \in \left(\frac{4}{7}, \frac{5}{7}\right), \\ 8 < 14\rho < 9 &\rightarrow \rho \in \left(\frac{8}{14}, \frac{9}{14}\right), \\ 10 < 19\rho < 11 &\rightarrow \rho \in \left(\frac{10}{19}, \frac{11}{19}\right), \\ 14 < 26\rho < 15 &\rightarrow \rho \in \left(\frac{14}{26}, \frac{15}{26}\right), \end{aligned}$$

where, we have only written the more relevant information obtained, which are given by the points of the orbit closer to the initial condition. So, we have shown that

$$0.5714 \approx \frac{4}{7} < \rho_{2,3}(34) < \frac{15}{26} \approx 0.5769.$$

In Figure 2, we represent several successive lower and upper approximations obtained while the orbit is turning around the oval. We plot around 600 steps, after skipping the first 50 ones. By taking 1000 points, we get

$$0.5761246 \approx \frac{338}{578} < \rho_{2,3}(34) < \frac{473}{821} \approx 0.5761267,$$

and after 3000 points,

$$0.57612457 \approx \frac{338}{578} < \rho_{2,3}(34) < \frac{1472}{2555} \approx 0.57612524.$$

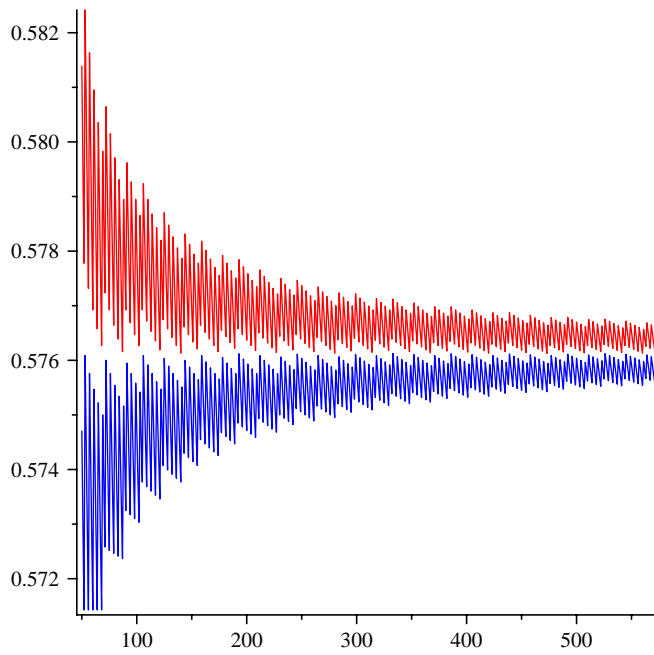


Figure 2. Lower and upper bounds for $\rho_{2,3}(34)$ obtained after following some points of the orbit starting at $(1, 1)$.

In fact, when we say that $\rho_{2,3}(34) \in (\rho_{\text{low}}, \rho_{\text{upp}})$, the value ρ_{low} is the upper lower bound obtained by following all the considered points of the orbit, and ρ_{upp} is the lowest upper bound. Notice that taking 1000 or 3000 points, we have obtained the same lower bound for $\rho_{2,3}(34)$.

Let us prove Theorem 3 by using the above approach.

Proof of Theorem 3. Consider $a = 1/2$, $b = 3/2$ and the three points

$$\mathbf{p}^1 = \left(\frac{149}{100}, \frac{173}{100} \right), \quad \mathbf{p}^2 = \left(\frac{3}{40}, \frac{173}{100} \right) \quad \text{and} \quad \mathbf{p}^3 = \left(\frac{1}{1000}, \frac{173}{100} \right).$$

Notice that

$$\begin{aligned} h_1 &:= V_{3/2, 1/2}(\mathbf{p}^1) = \frac{10,655,559}{1,288,850} \simeq 8.27, \\ h_2 &:= V_{3/2, 1/2}(\mathbf{p}^2) = \frac{9,328,327}{207,600} \simeq 44.93, \\ h_3 &:= V_{3/2, 1/2}(\mathbf{p}^3) = \frac{1,056,238,343}{346,000} \simeq 3052.71. \end{aligned}$$

Hence, $h_c < h_1 < h_2 < h_3$. By applying the algorithm described above, using 100 points of each orbit starting at each $\mathbf{p}^j, j = 1, 2, 3, \dots$ we obtain that

$$\rho_{3/2, 1/2}(h_1), \rho_{3/2, 1/2}(h_3) \in \left(\frac{3}{5}, \frac{59}{98} \right) \quad \text{and} \quad \rho_{3/2, 1/2}(h_2) \in \left(\frac{56}{93}, \frac{53}{88} \right).$$

Since $59/98 < 56/93$, we have proved that the function $\rho_{3/2,1/2}(h)$ has at least a local maximum in (h_1, h_3) . From the continuity of the rotation number function with respect to a , b and h , we notice that this result also holds for all values of a and b in the neighbourhood of $a = 1/2, b = 3/2$. \square

We believe that with the same method, it can be proved that a similar result to the one given in Theorem 3 holds for some maps $F_{c,b,a}$, but we have decided not to perform this study.

5.1 Some numerical explorations for $k = 2$

We start by studying in more detail the rotation number function $\rho_{3/2,1/2}(h)$ that we considered to prove Theorem 3. In this case, the fixed point is $\mathbf{p} \approx (1.493363282, 1.730133891)$ and $h_c = V_{b,a}(\mathbf{p}) \approx 8.267483381$. Moreover, $\rho_{b,a}(h_c) \approx 0.6006847931$. By applying our algorithm for approximating the rotation number, with 5000 points on each orbit, we obtain the results presented in Table 1. In Figure 3, we also plot the upper and lower bounds of $\rho_{3/2,1/2}(h)$ that we have obtained by using a wide range of values of h .

For other values of a and b , we obtain different behaviours. All the experiments are performed by starting at the fixed point $\mathbf{p} = (\bar{x}, \bar{y})$, and increasing the energy level by taking initial conditions of the form (x, \bar{y}) , by decreasing x to 0. With this process, we take orbits approaching the boundary of Q^+ , that is lying on level sets of $V_{b,a}$ with increasing energy. The step in the decrease of x (and, therefore, in the increase of h) is not uniform, and it has been manually tuned making it smaller in those regions where a possible non-monotonous behaviour could appear.

Consider the set of parameters $\Gamma = \{(a, b), \in [0, \infty)^2\}$, where notice that we also consider the boundaries $a = 0$ or $b = 0$, where the map $F_{b,a}$ is well defined. We already know that the rotation number function behaves equal at (a, b) and (b, a) . Moreover, we know perfectly its behaviour on the diagonal (a, a) (when $a < 1$ it is monotonous decreasing and when $a > 1$ it is monotonous increasing) and that $\rho_{1,1}(h) \equiv 4/5$ and $\rho_{0,0}(h) \equiv 2/3$. Hence, a good strategy for a numerical exploration can be to produce sequences of experiments using our algorithm by fixing some $a \geq 0$ and varying b . For instance, we obtain

- Case $a = 1/2$. For all the values of $b > 0$ considered, the rotation number function seems to tend to $3/5$ when h goes to infinity. Moreover, it seems:
 - monotone decreasing for $b \in \{1/4, 1\}$,
 - to have a unique maximum when $b \in \{7/5, 3/2\}$,
 - monotone increasing for $b \in \{2, 3\}$.

Table 1. Lower and upper bounds of the rotation number $\rho_{3/2,1/2}(h)$, for some orbits of $F_{3/2,1/2}$ starting at (x, \bar{y}) , where $\mathbf{p} = (\bar{x}, \bar{y})$.

Initial conditions (x, \bar{y})	Energy level h	$\rho_{\text{low}}(h)$	$\rho_{\text{upp}}(h)$
\bar{x}	$h_c \approx 8.2675$	≈ 0.6006848	≈ 0.6006848
1.3	8.3068	$(173/288) \approx 0.6006944$	$(2938/4891) \approx 0.6006951$
0.75	9.2747	$(1435/2388) \approx 0.6009213$	$(2087/3473) \approx 0.6009214$
0.3	14.7566	$(1548/2573) \approx 0.6016323$	$(2285/3798) \approx 0.6016324$
0.075	44.9347	$(657/1091) \approx 0.6021998$	$(2354/3909) \approx 0.6022001$
0.001	3052.75	$(2927/4867) \approx 0.6013972$	$(86/143) \approx 0.6013986$
5×10^{-6}	609716.07	$(1832/3049) \approx 0.6008527$	$(1409/2345) \approx 0.6008529$
5×10^{-256}	6.097×10^{255}	$(3/5) = 0.6$	$(2999/4998) \approx 0.6000400$

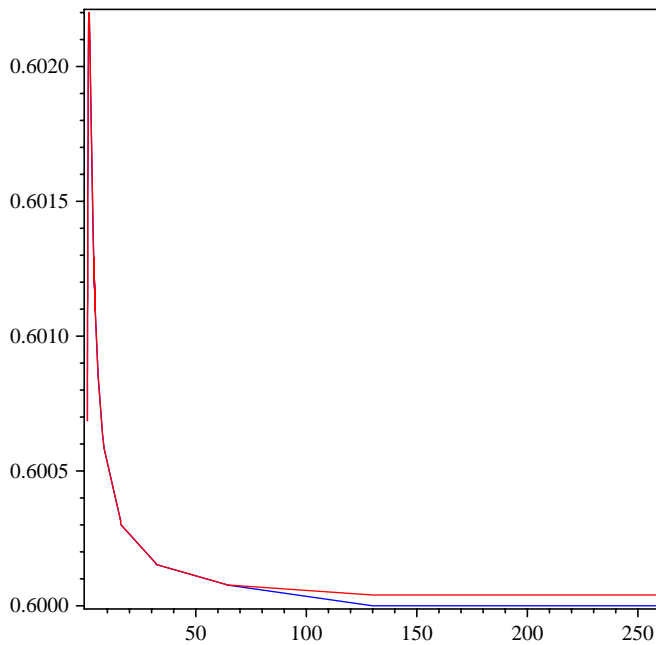


Figure 3. Lower and upper bounds for $\rho_{3/2, 1/2}(h)$. On the horizontal axis we represent $-\log_{10}(h)$ and, on the vertical axis, the value of the rotation number. Notice that for values of $-\log_{10}(h)$ smaller than 70, both values are indistinguishable in the figure.

- Case $a = 0$. For all the values of $b > 0$ considered, the rotation number function seems to tend to $5/8$ when h goes to infinity. Moreover, it seems;
 - monotone decreasing for $b \in \{1/10, 3/10, 1/2\}$,
 - to have a unique maximum when $b \in \{7/10, 3/4\}$,
 - monotone increasing for $b \in \{1, 5\}$.

The above results, together with some other experiments for other values of a and b , not detailed in this paper, indicate the existence of a subset of positive measure in Γ , where the corresponding rotation number functions seem to present a unique maximum. This subset probably separates two other subsets of Γ , one where $\rho_{b,a}(h)$ is monotonically decreasing to $3/5$, and another one where $\rho_{b,a}(h)$ increases monotonically to the same value. The ‘oscillatory subset’ seems to shrink to $(a, b) = (1, 1)$ when it approaches the line $a = b$ and seems to finish in one interval on each of the borders $\{a = 0\}$ and $\{b = 0\}$. Further analysis must be done in this direction in order to have a more accurate knowledge of the bifurcation diagram associated with the behaviour of $\rho_{b,a}$ on Γ .

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Notes

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2. Email: victor.manosa@upc.edu
3. Notice that given a map of the circle, there is an ambiguity between ρ and $1 - \rho$ when one considers its rotation number. So, in this paper the rotation number of the classical Lyness map for $a = 1$ is $4/5$, in other papers it is computed as $1/5$.

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